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Rigorous Proof of the High-Temperature Josephson Inequality for Critical Exponents

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I give a rigorous proof of the high-temperature Josephson inequality $d\nu \geqslant 2 - \alpha$, following the original ideas of Josephson. The proof is applicable to a class of models including the ferromagnetic Ising model and the φ^4 lattice field theory.

KEY WORDS: Josephson inequality; critical exponents; correlation inequalities; hyperscaling.

A decade ago, Josephson(1) gave a semirigorous proof of the critical-exponent inequalities[2]

$$d\nu \geqslant 2 - \alpha \tag{1}$$

and

$$d\nu' \geqslant 2 - \alpha' \tag{2}$$

These inequalities (and related ones(2–9)) are of considerable importance in the theory of critical phenomena(10); in particular, the hyperscaling conjecture(11,12) implies that they hold as *equalities*.

The general opinion(3,6) seems to be that Josephson's proof is "not rigorous and involve[s] assumptions whose validity is rather hard to judge even on an intuitive basis."(6) For this reason, alternative methods of proof have been sought: Stell(13) has proven a weakened version of (1), and the present author(9) has proven (2). What I should like to show here, however, is that Josephson's original method of proof can, with minor modifications, be made entirely rigorous—at least with regard to inequality (1)—and that

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[2] Critical exponents are defined in Eqs. (19)–(23) below.

the assumptions can be stated in an unambiguous form which makes clear their presumptive validity.

Consider a model of classical one-component spins on a d-dimensional lattice, defined formally by the partition function

$$Z = \int \exp\left(\frac{J}{2} \sum_{i,j} \alpha_{ij} \varphi_i \varphi_j \right) \prod_i d\nu(\varphi_i) \tag{3}$$

Here J is the inverse temperature and $\alpha_{ij} = \alpha_{ji}$ is the pair interaction; both are assumed nonnegative ("ferromagnetic"). The *a priori* single-spin measure $d\nu(\varphi)$ is assumed to be an even probability measure satisfying the hypotheses of the Lebowitz inequality[(14-17)3]; examples are the spin-1/2 Ising model $d\nu(\varphi) = \delta(\varphi^2 - 1)d\varphi$ and the φ^4 lattice field theory[(19,20)] $d\nu(\varphi) = \text{const} \times \exp(-a\varphi^2 - b\varphi^4)d\varphi$, $b > 0$. Inequalities (1) and (2) are undoubtedly true in much greater generality than this, but the present class of models suffices to show the method of proof.

Josephson's inequality is a lower bound on the specific heat (per lattice site)[4]

$$C_H = \frac{1}{4} J^2 \sum_{j,k,l} \alpha_{0j} \alpha_{kl} \langle \varphi_0 \varphi_j ; \varphi_k \varphi_l \rangle \tag{4}$$

Here $\langle A ; B \rangle$ denotes the truncated expectation $\langle AB \rangle - \langle A \rangle \langle B \rangle$, and I assume that the infinite-volume limit has been taken in such a manner as to yield a translation-invariant ergodic state ("pure phase"). Define also the susceptibility (per lattice site)[4]

$$\chi = \sum_j \langle \varphi_0 ; \varphi_j \rangle \tag{5}$$

and, for each $\phi > 0$, the correlation length of order ϕ,

$$\xi_\phi = \left(\chi^{-1} \sum_j |j|^\phi \langle \varphi_0 ; \varphi_j \rangle \right)^{1/\phi} \tag{6}$$

The proof of Josephson's inequality is based on the Schwarz inequality

[3] The Lebowitz inequality[(15,16)] states (among other things) that $\langle \varphi_i \varphi_j \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_j \rangle \langle \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_l \rangle - \langle \varphi_i \varphi_l \rangle \langle \varphi_j \varphi_k \rangle + 2\langle \varphi_i \rangle \langle \varphi_j \rangle \langle \varphi_k \rangle \langle \varphi_l \rangle \leqslant 0$. For the Lebowitz inequality to be valid, it suffices[(16,17)] that $d\nu(\varphi) = \exp[-V(\varphi)]d\varphi$, where V is even and differentiable, with V' convex on $(0, \infty)$; or that $d\nu$ be a limit of such measures. Also, the inequality holds for classical Ising models of arbitrary spin, by the "analog system" method of Griffiths.[(18)]

[4] I consider the specific heat C_H and the susceptibility χ to be *defined* by the stated expressions in terms of correlation functions. These values presumably coincide with the thermodynamic definitions as appropriate derivatives of the infinite-volume free energy density, but this equivalence is not trivial. It has, nevertheless, been proven in a number of cases; see, e.g., the appendix of Ref. 9.

for truncated expectations[5]

$$\langle B; B\rangle \geqslant \frac{\langle A; B\rangle^2}{\langle A; A\rangle} \tag{7}$$

Here we take

$$B = \sum_{i,j\in\Lambda} \alpha_{ij}\varphi_i\varphi_j \tag{8}$$

where Λ is a large but finite box of cardinality $|\Lambda|$; clearly

$$C_H \geqslant (J^2/4|\Lambda|)\langle B; B\rangle \tag{9}$$

by Griffiths' second inequality[(15)] and translation invariance. We need only choose a suitable A so as to make the numerator in (7) large and the denominator small. Since $\langle A; B\rangle \approx 2\ \partial\langle A\rangle/\partial J$ for large Λ,[6] we wish to choose an A whose expectation varies rapidly with temperature near the critical point. Following Josephson[(1)] with minor modifications, we let A be the fluctuation of the magnetization[7] in a cell of size L (to be chosen later):

$$A = \sum_{\substack{i,j\in\Lambda\\ |i-j|\leqslant L}} \varphi_i\varphi_j \tag{10}$$

To prove an upper bound for $\langle A; A\rangle$ is quite simple: by the Griffiths[(15)] and Lebowitz[(14–17)] inequalities and translation invariance,

$$\begin{aligned} 0 \leqslant \langle A; A\rangle &= \sum_{\substack{i,j,k,l\in\Lambda\\ |i-j|\leqslant L\\ |k-l|\leqslant L}} \langle \varphi_i\varphi_j; \varphi_k\varphi_l\rangle \\ &\leqslant \sum_{i\in\Lambda} \sum_{\substack{j,k,l\\ |k-l|\leqslant L}} \langle \varphi_i\varphi_j; \varphi_k\varphi_l\rangle \\ &\leqslant 2\sum_{i\in\Lambda} \sum_{\substack{j,k,l\\ |k-l|\leqslant L}} \langle \varphi_i\varphi_k\rangle\langle \varphi_j\varphi_l\rangle \\ &\leqslant \text{const} \times |\Lambda| L^d \chi^2 \end{aligned} \tag{11}$$

[5]Note that $\langle A; B\rangle$ is a symmetric bilinear form in A and B, and it is *positive* because $\langle A; A\rangle = \langle A^2\rangle - \langle A\rangle^2 \geqslant 0$ by the ordinary Schwarz inequality. Hence the usual proof of the Schwarz inequality goes through entirely unchanged. I learned this simple observation from Antti Kupiainen.

[6]If the requisite "fluctuation–dissipation theorem" holds; see the appendix of Ref. 9.

[7]Since we are interested here in the region just *above* the critical temperature, where the *average* magnetization is zero, no such term need be subtracted from (10). But see the remarks at the end of this paper concerning a possible extension to the low-temperature critical exponents [Eq. (25) ff].

(Here I have anticipated the fact that the average magnetization vanishes for $J < J_c$, so that, e.g., $\langle \varphi_i \varphi_k \rangle = \langle \varphi_i ; \varphi_k \rangle$.)

The lower bound on $\langle A ; B \rangle$ is slightly (but not much) more subtle. By the Griffiths inequality,[8]

$$\begin{aligned} L^{\phi} \sum_{\substack{i,j,k,l \in \Lambda \\ |i-j|>L}} \alpha_{kl} \langle \varphi_i \varphi_j ; \varphi_k \varphi_l \rangle &\leqslant \sum_{\substack{i,j,k,l \in \Lambda \\ |i-j|>L}} |i-j|^{\phi} \alpha_{kl} \langle \varphi_i \varphi_j ; \varphi_k \varphi_l \rangle \\ &\leqslant \sum_{\substack{i \in \Lambda \\ j,k,l}} |i-j|^{\phi} \alpha_{kl} \langle \varphi_i \varphi_j ; \varphi_k \varphi_l \rangle \\ &= 2|\Lambda| \frac{\partial}{\partial J} \left(\chi \xi_{\phi}^{\phi} \right) \end{aligned} \tag{12}$$

where the last equality in (12) follows from a suitable fluctuation–dissipation theorem (see footnote 6) (and translation invariance). Hence

$$|\Lambda|^{-1} \sum_{\substack{i,j,k,l \in \Lambda \\ |i-j|>L}} \alpha_{kl} \langle \varphi_i \varphi_j ; \varphi_k \varphi_l \rangle \leqslant 2 \left(\frac{\xi_{\phi}}{L} \right)^{\phi} \left(\frac{\partial \chi}{\partial J} + \phi \frac{\chi}{\xi_{\phi}} \frac{\partial \xi_{\phi}}{\partial J} \right) \tag{13}$$

On the other hand, as $\Lambda \to \infty$ we have

$$\begin{aligned} |\Lambda|^{-1} \sum_{i,j,k,l \in \Lambda} \alpha_{kl} \langle \varphi_i \varphi_j ; \varphi_k \varphi_l \rangle &\to \sum_{j,k,l} \alpha_{kl} \langle \varphi_0 \varphi_j ; \varphi_k \varphi_l \rangle \\ &= 2 \frac{\partial \chi}{\partial J} \end{aligned} \tag{14}$$

The convergence is elementary,[9] and the equality is a fluctuation–dissipation theorem.[6] It follows that

$$\liminf_{\Lambda \to \infty} |\Lambda|^{-1} \langle A ; B \rangle \geqslant 2 \frac{\partial \chi}{\partial J} \left[1 - \left(\frac{\xi_{\phi}}{L} \right)^{\phi} \left(1 + \phi \frac{\chi}{\xi_{\phi}} \frac{\partial \xi_{\phi} / \partial J}{\partial \chi / \partial J} \right) \right] \tag{15}$$

We shall take

$$L = \left[2 \left(1 + \phi \frac{\chi}{\xi_{\phi}} \frac{\partial \xi_{\phi} / \partial J}{\partial \chi / \partial J} \right) \right]^{1/\phi} \xi_{\phi} \tag{16}$$

so that

$$\liminf_{\Lambda \to \infty} |\Lambda|^{-1} \langle A ; B \rangle \geqslant \frac{\partial \chi}{\partial J} \tag{17}$$

[8] A similar argument has been used by Fisher[(6)]; cf. Eq. (39) ff.

[9] Use translation invariance to rewrite the left side of (14) as $\sum_{j,k,l} c_{jkl}(\Lambda) \alpha_{kl} \langle \varphi_0 \varphi_j ; \varphi_k \varphi_l \rangle$, where the coefficients $c_{jkl}(\Lambda)$ depend on the geometry of Λ. Now $c_{jkl}(\Lambda) \leqslant 1$ and $c_{jkl}(\Lambda) \to 1$ as $\Lambda \to \infty$; so the convergence follows from the dominated convergence theorem (and Griffiths' inequality), provided that the sum is finite.

Combining (7), (9), (11), and (17), we conclude that

$$C_H \geqslant \text{const} \times J^2(\partial\chi/\partial J)^2 L^{-d}\chi^{-2} \tag{18}$$

with L given by (16).

For lattice dimension $d \geqslant 2$, there is a phase transition at J equal to some critical value J_c.[21] The critical exponents α, γ, and ν_ϕ are usually defined[10] by assuming that[10]

$$C_H \sim (J_c - J)^{-\alpha} \tag{19}$$

$$\chi \sim (J_c - J)^{-\gamma} \tag{20}$$

$$\xi_\phi \sim (J_c - J)^{-\nu_\phi} \tag{21}$$

as $J \uparrow J_c$. Here we need to make a slightly stronger assumption, namely,

$$\frac{\partial\chi}{\partial J} \sim (J_c - J)^{-(\gamma+1)} \tag{22}$$

$$\frac{\partial\xi_\phi}{\partial J} \sim (J_c - J)^{-(\nu_\phi+1)} \tag{23}$$

It then follows that L/ξ_ϕ is bounded as $J \uparrow J_c$, so (18) implies immediately that

$$d\nu_\phi \geqslant 2 - \alpha \qquad \text{for all } \phi > 0 \tag{24}$$

This is the precise version of (1). [One can also define an exponent ν associated with the "true" (exponential) correlation length ξ; if reflection positivity[22] holds, then $\nu \geqslant \nu_\phi$ for all ϕ,[9] so that (1) also holds as is.]

Unfortunately, the foregoing argument does not immediately generalize to prove the low-temperature Josephson inequality (2). In the presence of a nonzero spontaneous magnetization

$$M = \langle \varphi_0 \rangle \tag{25}$$

Josephson[1] would replace the definition (10) by

$$A' = \sum_{\substack{i,j \in \Lambda \\ |i-j| \leqslant L}} (\varphi_i - M)(\varphi_j - M) \tag{26}$$

Then it is not hard to show, using the version of the Lebowitz inequality appropriate to $M \neq 0$, that

$$0 \leqslant \langle A'; A' \rangle \leqslant \text{const} \times |\Lambda| \left[L^d\chi^2 + L^{2d}M^2\chi \right] \tag{27}$$

[10]More precisely, $\alpha = \lim_{J \uparrow J_c} [-\log C_H / \log(J_c - J)]$, and similarly for γ and ν_ϕ. Note, by the way, that α is the exponent for the full specific heat, *not* for its singular part. Therefore, $\alpha \geqslant 0$ unless C_H *vanishes* at the critical point (which is extremely unlikely).

This replaces (11). The trouble is with obtaining the lower bound on $|\langle A'; B\rangle|$: since $\langle \varphi_i; \varphi_j; \varphi_k \varphi_l\rangle$ does not have any definite sign, the argument analogous to (12) does not work for A' (even though one does expect the analog of (13) to be *true*). The difficulty is similar to Fisher's[(6)] inability to prove the critical-exponent inequality $\gamma' \leqslant (2-\eta)\nu'_{\phi}$.

Of course, inequality (2) [and a *weakened* version of the analog of (24)] has been proven by different means.[(9)] But it would be pleasant to find a rigorous version of Josephson's original proof. After all, his idea was better than we had thought.

NOTE ADDED IN PROOF

A heuristic discussion of the Josephson inequality is also given by Glimm and Jaffe[(23)].

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